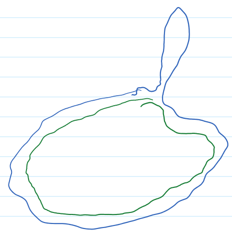
Going further: how fast does it converge?

Want something like $\mathrm{dist}(\gamma^n, \gamma) \lesssim n^{-\alpha}$ for some α – polynomial rate of convergence.

Axiomatic approach: (B – Richards)

Larissa Richards



Setup: Ω – simply connected domain,
Ω^n – interior $\frac{1}{n}$-approximations, Jordan.
$(\forall\ z \in \partial\Omega^n,\ \mathrm{dist}(z, \partial\Omega) < \frac{1}{n};\ \Omega^n \subset \Omega)$.

Typical example: lattice approximation
Let $a, b \in \widehat{\partial\Omega}$, let $\varphi : (\Omega, a, b) \to (\mathbb{H}, 0, \infty)$ - conformal.

Select $a^n, b^n \in \partial\Omega^n$ such that $\mathrm{dist}_{\mathbb{C}}(a, a^n) \to 0$
$\mathrm{dist}_{\mathbb{C}}(b, b^n) \to 0$ polynomially fast

So $\exists\, \varphi^n$: 1) $\varphi^n \to \varphi$ uniformly on compacts.
conformal 2) $\varphi^n(a^n) \to \varphi(a)$, $\varphi^n(b^n) \to \varphi(b)$

Let γ^n – random curves in Ω^n from a^n to b^n.
γ – random curve in Ω from a to b.

$\bar\gamma^n := \varphi^n(\gamma^n)$, $\bar\gamma := \varphi(\gamma)$.

We assume that γ^n, γ (and, equivalently, $\bar\gamma^n, \bar\gamma$) are a.s. Löewner curves.

Let W_t^n, W_t be driving functions of $\bar\gamma^n, \bar\gamma$;

$g_t^n : \mathbb{H} \setminus \bar\gamma^n[0,t] \to \mathbb{H}$
$g_t : \mathbb{H} \setminus \bar\gamma[0,t] \to \mathbb{H}$ } corresponding Löwner chains.

$\Omega_t^n := (\varphi^n)^{-1}(\mathbb{H} \setminus \bar\gamma^n([0,t]) = \Omega^n \setminus \gamma^n([0,t])$
$\Omega_t := \varphi^{-1}(\mathbb{H} \setminus \bar\gamma[0,t]) \underset{\text{unbounded component}}{=} \Omega \setminus \gamma([0,t])$ } component with b^n at the boundary.

We will use multivariable observables, so let us introduce some notations:

m, ℓ – fixed, $\hat{v}^n := (v_1^n, \dots, v_m^n; c_1^n, \dots, c_\ell^n)$ be a collection of m interior points and ℓ boundary points. $V^{m,\ell}(\Omega^n)$ – the set of all such collections.

(if we think about lattice approximations, we only need to take v_j^n, c_j^n to be vertices)

Also define

$$\hat{\varphi}(\hat{v}^n) := (\varphi(v_1^n), \dots, \varphi(v_m^n), \varphi(c_1^n), \dots, \varphi(c_\ell^n)) \in \mathbb{H}^{m+\ell}$$

$$\hat{\varphi}^n(\hat{v}^n) := (\varphi^n(v_1^n), \dots, \varphi^n(v_m^n), \varphi^n(c_1^n), \dots, \varphi^n(c_\ell^n)) \in \mathbb{H}^m \times \mathbb{R}^\ell$$

(well defined for c_j^n since φ^n is extendable to the boundary).

Let $\gamma^n_{(\Omega, a, b)}$ be a family of random curves, indexed by simply connected Ω, $a, b \in \partial\Omega$, such that $\forall n, \Omega, a, b$ $\gamma^n_{(\Omega,a,b)}$ is a.s. a curve in Ω^n from a^n to b^n.

Assume that it satisfies Domain Markov property:

$$\left(\gamma^n_{(\Omega,a,b)}[t,\infty) \,\middle|\, \gamma^n_{(\Omega,a,b)}[0,t]\right) = \gamma^n_{(\Omega \setminus \gamma^n[0,t],\, \gamma^n(t),\, b)}.$$

Assume also that $\exists\, m, \ell$: $\exists\, 0 < T = T(\hat{v}) = T(\Omega, a, b, \varphi, \hat{v})$ – stopping time, continuous in $\hat{v}$, $s \in (0,1)$, n_0, such that $\forall n > n_0$:

1) (Discrete almost martingale observable)

$\exists\ H^n_{(\Omega^n, a^n, b^n)} : V^{m,\ell}(\Omega^n) \to \mathbb{C}$, such that

$$\left| H^n_{(\Omega^n_t, \gamma^n(t), b^n)}(\hat{v}) - E\left(H^n_{(\Omega^n_{t'}, \gamma^n(t'), b^n)} \,\middle|\, \Omega^n_t \right) \right| \le n^{-s}$$

$\forall \hat{v} \in V^{m,\ell}(\Omega)$, $0 \le t \le t' \le T(\hat{v})$

2) (Limiting observable)

$\exists\, h : \mathbb{H}^m \times \mathbb{R}^\ell \to \mathbb{R}$, which satisfies

$$\sum_{j=1}^m \left(\frac{1}{(x^j)^2 + (y^j)^2} \left(x^j \frac{\partial h}{\partial x^j} - y^j \frac{\partial h}{\partial y^j} \right) \right) + \sum_{j=1}^{\ell} \frac{1}{c^j} \frac{\partial h}{\partial c^j} +$$

$$\frac{\kappa}{2} \left(\sum_{j=1}^m \frac{\partial^2 h}{(\partial x^j)^2} + \sum_{j=1}^{\ell} \frac{\partial^2 h}{(\partial c^j)^2} \right) = 0$$

(this is the condition which makes $h(\mathbb{H} \setminus \gamma[0,t])$ a martingale for γ being SLE_κ).

We also need h to be non-degenerate.

3) (Discrete and continuous observable are polynomially close).

$$|H^n_{(\Omega_t,a,b)}(\hat{v}) - h(\hat{\varphi}^n_t(\hat{v}))| \le n^{-s}$$

for all $\hat{v}$ with $\mathrm{dist}(v_j^n, \partial\Omega) \ge n^{-s}$ $\forall\, 1 \le j \le m$.
and $t \le T(\hat{v})$.

4) $\gamma^n_{(\Omega,a,b)}$ satisfy KS-condition.

Theorem (B-Richards).

Let $\gamma^n_{(\Omega,a,b)}$ satisfy 1)-4) for some $\kappa < 8$.

Then $\exists$ a coupling of $\gamma^n_{(\Omega,a,b)}$ with BM $B(t)$ such that, for $\bar{\gamma}$ being driven by $\sqrt{\kappa}B(t)$ (SLE$_\kappa$!)

$$\mathbb{P}\left(\sup_{t \le T} |\bar{\gamma}^n(t) - \bar{\gamma}(t)| > n^{-u}\right) < n^{-u} \quad \text{(Prokhorov distance)}$$

$\forall n \ge N$. $u = u(s)$

$N = N(s, n_0, T)$.

Moreover, if Ω, Ω_n are α-Hölder
(for conformal $\psi : \mathbb{D} \to \Omega$, $|\psi'(z)| \le \frac{C}{(1-|z|)^{1-\alpha}}$)

Then
for some $\lambda = \lambda(s, \alpha)$ we have

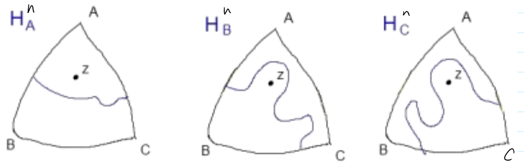
$$P\left(\sup_{t \le T} |\gamma^n_t - \varphi^{-1}(\bar{\gamma}_t)| > n^{-\lambda}\right) < n^{-\lambda}, \quad n \ge N.$$

SLE$_\kappa$ in (Ω, a, b)

An application: critical Percolation on hexagonal lattice.

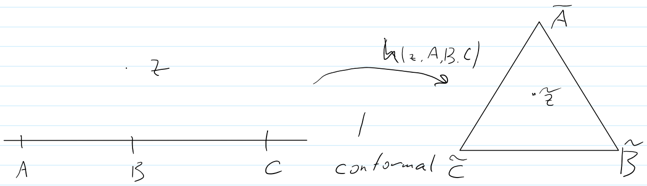
The observable:

Cardy-Smirnov observable:



$\tau := e^{\frac{2\pi i}{3}}$

$H^n(z, A, B, C; \Omega) := H_A^n + \tau H_B^n + \tau^2 H_C^n.$



Condition 4: KS condition is satisfied.

Condition 3: (Discrete is close to continuous).


Lincoln Chayes


Helen Lei

Theorem (B-Lei-Chayes)

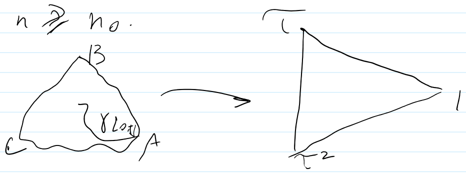
Let Ω be a domain, $A, B, C \in \widehat{\partial\Omega}$ - prime ends, $(\Omega^n, A^n, B^n, C^n)$ - interior approximation by $\frac{1}{n}$-hexagonal lattice. Let γ^n be an interface from A^n to C^n on the lattice, $T := \inf\{t : |\gamma^n(t) - C^n| < \Delta\}$ (fixed Δ)

Then $\exists\ s$ (domain independent!) and $n_0 = n_0(\Omega, A, B, C, \Delta)$ such that

$$\left| H^n(z, \gamma^n(t), B, C; \Omega \setminus \gamma^n([0,t])) - h_t(z) \right| \leq n^{-s}$$

for $n \geq n_0$.

$h_t(z)$:



Condition 2: direct computation (or can be seen from almost martingale property, convergence, and the fact that all SLE_6 martingales satisfy Condition 2).

Condition 1.

As in previous part,

$$H^n_{(\Omega^n_t,\gamma^n(t),b^n)}(\hat{v}) = E\left(H^n_{(\Omega^n_{t'},\gamma^n(t'),b^n)} \,\middle|\, \Omega^n_t\right) \text{ for } t<t' \quad (\notin \partial\Omega_t).$$

$\vdots$

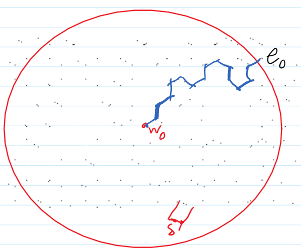
Where else does it work?

1) Loop Erased Random walk:
(Benes, Viklund, Kozdron)

Christian Benes

Michael Kozdron

Fredrik Viklund



LERW$_\delta$.

We'll need radial version here.

Observable: Let $e_0 \in \partial\Omega$, $w_0 \in \Omega$ - fixed.

$$H^\delta(z) = \frac{P(RW_\delta \text{ started at } z \text{ exits } \Omega \text{ at } e_0)}{P(RW_\delta \text{ started at } w_0 \text{ exits } \Omega \text{ at } e_0)}$$

$h(z) = \frac{1}{\pi}\cdot\frac{y}{x^2+y^2}$ - Poisson kernel.

Limiting curve: SLE_2 (radial)

FK Ising: observable: fermionic observable.

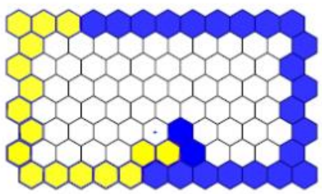
$$h = \frac{1}{\sqrt{\pi z}}$$

In domain: if $\varphi: (\Omega, a, b) \to (\mathbb{R}\times(0,1), -\infty, \infty)$ - conformal,

$$h = (\varphi')^{1/2}$$

Limiting curve: $SLE_{8/3}$

Harmonic Explorer:



Almost the same rules as the Exploration Process:

- if the path encounters yellow face, turn right
- if the path encounters blue face, turn left.

- if the path encounters an unexplored face, start a random walk at this face, and run it until it hits explored part. Color in the color that the random walk hits.

Observable: Probability that path will go to the left of the point = $\mathbb{P}$(RW from z hits a blue face) — discrete harmonic measure of the blue part.

Continuous counterpart: harmonic measure of the arc (a,b)

In $\mathbb{H}$: $h(z) = 1 - \frac{1}{\pi} \operatorname{Arg} z$.

Limiting curve: SLE_4.

Dmitry Chelkak

Polynomial Rate of Convergence for HE; B-Chelkak-Richards.

On the proof for the framework:

Step 1. From convergence of observable to convergence of driving function.

For this step, we just need conditions 1) – 3) (KS is not required)

To simplify the notations, let h be a <u>one</u>-variable observable.

Fix w_0 far enough from the boundary. Let $W^n(t)$ be the driving function of γ^n

From 1) (almost martingale for discrete) and 3) (continuous close to discrete)

we get that for small t

$$E\left(h(g_{t'}(\varphi(w_0)) - W^n(t')) \,\middle|\, \gamma[0,t]\right) = h(g_t(\varphi(w_0)) - W^n(t)) + O(n^{-s})$$

Now expand $h(g_{t+\varepsilon}(\varphi(w_0)) - W^n(t+\varepsilon))$ near $(g_t(\varphi(w_0)) - W^n(t))$

gives   we use 2) here

$$O(n^{-s}) = A_t\left(E\left((W^n(t+\varepsilon) - W^n(t))^2 - \kappa\varepsilon\right)\right) + B_t\, E\left(W^n(t+\varepsilon) - W^n(t)\right)$$

By taking two nearby w_0, w_1 and using nondegeneracy we get

$$E\left(W^n(t+\varepsilon) - W^n(t)\right)^2 = \kappa\varepsilon + O(n^{-s})$$
$$E\left(W^n(t+\varepsilon) - W^n(t)\right) = O(n^{-s}).$$

Anatoliy Skorokhod (1930-2011)

Theorem. (Skorokhod embedding Theorem)

Let $(M_k)_{k\le K}$ be a martingale with $M_0 = 0$ a.s, $\|M_{k+1} - M_k\|_\infty < \delta$

Then $\exists$ $0 = T_0 < T_1 < \dots < T_K$ – stopping times for the standard BM $B(t)$ such that

1) The joint law of $(M_0, \dots, M_K)$ is the same as

$(B_{T_0}, B_{T_1}, \ldots, B_{T_k})$.

2) $E(T_{k+1} - T_k \mid B[0,T_k]) = E\left((B(T_{k+1}) - B(T_k))^2 \mid B[0,T_k] \right)$

3) $T_{k+1} \leq \inf\{t \geq T_k : |B(t) - B(T_k)| \geq \delta\}$.

The theorem gives the existence of a coupling

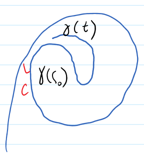
of $B(\kappa t)$ and $W^n(t)$ such that $P(\sup |W^n(t) - B(\kappa t)| > n^{-r}) < n^{-r}$.

Step 2. From the convergence of driving function to the convergence of curves.

Polynomial convergence of driving function $\Rightarrow$ polynomial convergence of curves.

Not true in general!

Need Tip Structure Modulus (Viklund). Define for $\gamma \in \mathbb{H}$.



Let C be a crosscut in Ω_t,

diam $C < \delta$, C separates $\gamma(t)$ from ∞.

Let $s_0 = \sup\{s > 0 : \gamma[s,t] \cap \bar{C} \neq \emptyset\}$

$\eta_{tip}(\delta) = \sup_C \operatorname{diam} \gamma[s_0, t]$.

Lemma (Viklund).

Assume that $\gamma, \tilde{\gamma}$ two Löwner curves,

$W_t, \tilde{W}_t$ — corresponding driving functions,

$f_t, \tilde{f}_t$: Löwner maps.

Let $\rho > 1$. Assume that $\exists\ \beta < 1$, $0 < r < 1$, $0 < p < \frac{1}{\rho}$,

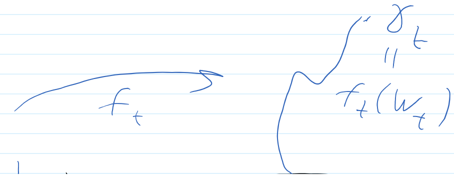
such that

1) (Driving terms are close)

$|W(t) - \tilde{W}(t)| \leq \varepsilon$.

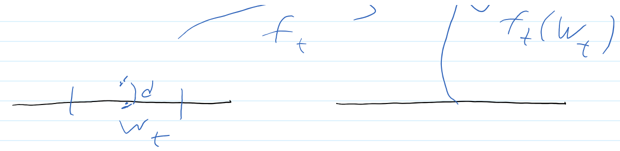
2) $\eta^{\tilde{\gamma}}_{tip}(\varepsilon^p) \leq c\,\varepsilon^{pr}$ for some $c > 0$.

3) $\sup |f_t'(W(t) + id)| \leq c' d^{-\beta}$ for all $d \leq \varepsilon^p$,



2) $\eta_{tip}(\varepsilon^{\ldots}) \le c\,\varepsilon^{\ldots}$ for some $c>0$.

3) $\sup |f_t'(W(t)+id)| \le c' d^{-\beta}$ for all $d \le \varepsilon^p$, and some $c'>0$.



Then $\exists\, c'' = c''(\beta, r, p, c, c')$ such that

$$\sup |\gamma(t) - \tilde{\gamma}(t)| \le c'' \max\left(\varepsilon^{p(1-\beta)r}, \varepsilon^{(1-p\rho)r}\right)$$

polynomial

How to use the Lemma?

Part I (Viklund) SLE_κ satisfies condition 3): $\exists\, c$:

$$\mathbb{P}\left(\forall d \le \varepsilon^p, \sup |f_s'(W_t + id)| \le c\, d^{-\beta}\right) \ge 1 - c\,\varepsilon^{pq},$$

where $1 \ge \beta \ge \max\left(0, \frac{4(\kappa\sqrt{8+\kappa} - (4-\kappa))}{(4+\kappa)^2}\right)$,

$q = q(\beta)$ (explicit!).

Part II (B- Richards)

KS condition implies condition 2)

Lemma Let (γ) satisfies KS condition

Then for some $\beta>0$, $\varepsilon>0$, $c, c'>0$, if $r = c R^{1+\varepsilon}$,

then $\mathbb{P}(\eta_{tip}(r) > R) < C R^{\beta}$.

(all constants depend only on p in KS)

Corollary. If γ^n satisfies (KS) condition

Then $\exists\, C, p, u, \alpha$:

$$\mathbb{P}\left(\eta_{tip}^{\gamma^n}(n^{-p}) > C n^{-pu}\right) < C n^{-\alpha}$$

Just take $R = C n^{-pu}$ in Lemma.
$r = n^{-p}$